GOVERNMENT OF ANDHRA PRADESH

COMMISSIONERATE OF COLLEGIATE EDUCATION

INVERSE LAPLACE TRANSFORMATION

HEAVISIDE’S EXPANSION FORMULA

MATHEMATICS

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Inverse Laplace Transformation – Heaviside's Expansion Formula

Learning Objectives

Students will be able to recognize the following properties of the Inverse Laplace Transformation.

- Understand the Heaviside's Expansion formula in Inverse Laplace transforms.
- Get an idea about degree of a polynomial.
- Get the knowledge of application of Laplace transformations.
- Students will be able to know the use of Inverse Laplace transform in system modeling, digital signal processing, etc.,

Heaviside's Expansion Formula

Theorem: Let F(p) and G(p) be two polynomials in p, where F(p) has degree less than that of G(p). If G(p) has n distinct zeros α_i, i = 1, 2, 3, … .

$$i.e\ G(p) = (p-\alpha_1)(p-\alpha_2)(p-\alpha_3)\ldots.(p-\alpha_n)\ then$$

$$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)}\ e^{\alpha_r t.}$$

Proof: Since F(p) is a polynomial of degree less than that of G(p) and G(p) has n distinct zeros α_i, i = 1, 2, 3, … .

$$\frac{F(p)}{G(p)} = \frac{F(p)}{(p-\alpha_1)(p-\alpha_2)(p-\alpha_3)\ldots.(p-\alpha_n)} = \frac{A_1}{(p-\alpha_1)} + \frac{A_2}{(p-\alpha_2)} + \cdots + \frac{A_n}{(p-\alpha_n)}$$

Multiplying both sides by $(p-\alpha_i)$ and taking limits as $p \to \alpha_r$, we have

$$A_r = \lim_{p\to\alpha_r} \frac{F(p)(p-\alpha_r)}{G(p)} = F(\alpha_r) \lim_{p\to\alpha_r} \frac{(p-\alpha_r)}{G(p)} = F(\alpha_r) \lim_{p\to\alpha_r} \frac{1}{G'(\alpha_r)}$$

$$\therefore \frac{F(p)}{G(p)} = \frac{F(\alpha_1)}{G'(\alpha_1)} \frac{1}{p-\alpha_1} + \frac{F(\alpha_2)}{G'(\alpha_2)} \frac{1}{p-\alpha_2} + \cdots + \frac{F(\alpha_r)}{G'(\alpha_r)} \frac{1}{p-\alpha_r} + \cdots + \frac{F(\alpha_n)}{G'(\alpha_n)} \frac{1}{p-\alpha_n}$$

$$\therefore L^{-1}\left\{\frac{F(p)}{G(p)}\right\} =$$

$$\frac{F(\alpha_1)}{G'(\alpha_1)} L^{-1}\left\{\frac{1}{p-\alpha_1}\right\} + \frac{F(\alpha_2)}{G'(\alpha_2)} L^{-1}\left\{\frac{1}{p-\alpha_2}\right\} + \cdots + \frac{F(\alpha_r)}{G'(\alpha_r)} L^{-1}\left\{\frac{1}{p-\alpha_r}\right\} + \cdots +$$

$$\frac{F(\alpha_n)}{G'(\alpha_n)} L^{-1}\left\{\frac{1}{p-\alpha_n}\right\}$$

$$\therefore L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)}\, e^{\alpha_r t}$$

Ex 1: Apply Heaviside's expansion formula to find $L^{-1}\left\{\frac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\}$

Sol: Let F(p) =6p^2 + 22p + 18 and G(p)= p^3 + 6p^2 + 11p + 6 = (P + 1)(P + 2)(p + 3)

Then $G'(p) = 3p^2 + 12p + 11$

Therefore G(p) has three distinct zeros α_1 = -1 , α_2 = -2, α_3 = -3

By Heaviside's expansion formula, we have

$$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = L^{-1}\left\{\frac{6\mathrm{p}^2+22\mathrm{p}+18}{\mathrm{p}^3+6\mathrm{p}^2+11\mathrm{p}+6}\right\} = \sum_{r=1}^{3}\frac{F(\alpha_r)}{G'(\alpha_r)}\ e^{\alpha_r t}$$

$$= \frac{F(-1)}{G'(-1)}e^{-t} + \frac{F(-2)}{G'(-2)}e^{-2t} + \frac{F(-3)}{G'(-3)}e^{-3t}$$

$$= \frac{2}{2}e^{-t} + \frac{-2}{-1}e^{-2t} + \frac{6}{2}e^{-3t}$$

$$= e^{-t} + 2e^{-2t} + 3e^{-3t}$$

$$\therefore L^{-1}\left\{\frac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\} = e^{-t} + 2e^{-2t} + 3e^{-3t}$$

Ex 3: Apply Heaviside's expansion formula to find $\boldsymbol{L^{-1}\left\{\frac{2p^2-6p+11}{p^3-6p^2+11p-6}\right\}}$

Sol: Let $F(p) = 2p^2 - 6p + 11$ and $G(p) = p^3 - 6p^2 + 11p - 6$

$= (P - 1)(P - 2)(p - 3)$

Then $G'(p) = 3p^2 - 12p + 11$

Therefore G(p) has three distinct zeros $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$

By Heaviside's expansion formula, we have

$$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = L^{-1}\left\{\frac{6p^2-6p+11}{p^3-6p^2+11p-6}\right\} = \sum_{r=1}^{3} \frac{F(\alpha_r)}{G'(\alpha_r)}\ e^{\alpha_r t}$$

$$= \frac{F(1)}{G'(1)}e^{t} + \frac{F(2)}{G'(2)}e^{2t} + \frac{F(3)}{G'(3)}e^{3t}$$

$$= \frac{7}{2}e^{t} + \frac{7}{-1}e^{2t} + \frac{11}{2}e^{3t}$$

$$\therefore L^{-1}\left\{\frac{2p^2-6p+11}{p^3-6p^2+11p-6}\right\} = \frac{7}{2}e^{t} - 7e^{2t} + \frac{11}{2}e^{3t}$$

Ex 4: Apply Heaviside's expansion formula to find $L^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\}$

Sol: Let f(p) = 3p + 1 and G(p) = (p - 1) (p^2 + 1)

and $G'(p) = 3p^2 - 2p + 1$

Therefore G(p) has three distinct zeros $\alpha_1 = 1$, $\alpha_2 = i$, $\alpha_3 = -i$

By Heaviside's expansion formula, we have

$$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = L^{-1}\left\{\frac{3\mathrm{p}+1}{(\mathrm{P}+1)(\mathrm{p}^2+1)}\right\} = \sum_{r=1}^{3}\frac{F(\alpha_r)}{G'(\alpha_r)}\ e^{\alpha_r t}$$

$$= \frac{F(1)}{G'(1)}e^{t} + \frac{F(i)}{G'(i)}e^{it} + \frac{F(-i)}{G'(-i)}e^{-it}$$

$$= \frac{4}{2}e^{t} + \frac{(3i+1)}{-2-2i}e^{it} + \frac{-3i+1}{-2+2i}e^{-it}$$

$$= 2e^{t} + \frac{(3i+1)(1-i)}{-2(1+i)(1-i)}e^{it} + \frac{(1+i)(-3i+1)}{2(1-i)(1+i)}e^{-it}$$

$$= 2e^{t} + \frac{(2i+4)}{4}e^{it} + \frac{(2i-4)}{4}e^{-it}$$

$$= 2e^{t} - \frac{i}{2}(e^{it}-e^{-it}) - (e^{it} + e^{-it})$$

$$= 2e^{t} - \frac{i}{2}(2i\sin t) - (2\cos t)$$

$$= 2e^{t} - \sin t - 2\cos t$$

$$\therefore L^{-1}\left\{\frac{3\mathrm{p}+1}{(\mathrm{P}+1)(\mathrm{p}^2+1)}\right\} = 2e^{t} - \sin t - 2\cos t$$

Practice Questions:

Using Heaviside's expansion formula, find

1. $L^{-1}\left\{\frac{p^2-6}{p^3+4p^2+3p}\right\}$ 2. $L^{-1}\left\{\frac{2p^2+5p-4}{p^3+p^2-2p}\right\}$ 3. $L^{-1}\left\{\frac{p-1}{(p+3)(p^2+2p+2)}\right\}$

Reference Books:

- Integral transforms by Ian Sneddon, Mc Graw hill publications.
- Introduction to Applied Mathematics by Gilbert Strang, Cambridge Press
- Laplace and Fouries transforms by Dr.J.K. Goyal and K.P. Guptha, Pragathi Prakashan, Meerut.
- The Laplace Transform: Theory and Applications by Joel L. Schiff
- Student's Guide to Laplace Transforms (Student's Guides) by Daniel Fleisch.


Thank you!